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Computing the elasticity of a Krull monoid☆

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Abstract

If S is a Krull monoid with finitely generated divisor class group such that only finitely many divisor classes of S contain prime divisors, then we construct an algorithm to compute the elasticity of S. © 2001 Elsevier Science Inc. All rights reserved.

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1. Introduction

The study of factorization properties of integral domains and monoids has been a topic of much recent literature (see [1] and the articles contained within). If S is an atomic integral domain (i.e., a domain in which every nonzero nonunit can be factored as a product of irreducible elements) with set $\mathscr{A}(S)$ of irreducible elements,

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then a central focus of this literature is the *elasticity* of S defined by

$$\rho(S) = \sup\left\{ \frac{n}{m} \,\middle|\, \alpha_1 \cdots \alpha_n = \beta_1 \cdots \beta_m \text{ where each } \alpha_i \text{ and } \beta_j \in \mathscr{A}(S) \right\}.$$

Elasticity was introduced by Valenza in [17] (and using different terminology by Michael and Steffan in [13]) and has been studied extensively by D.D. Anderson and D.F. Anderson [2], Halter-Koch [12], Chapman and Smith [8], and D.D. Anderson et al. [3]. A survey of recent works in elasticity can be found in [4].

Building on the original work of Valenza [17] for rings of algebraic integers, most of the works mentioned above consider elasticity in the case where S is a Krull domain or monoid. In a recent paper of Rosales et al. [16], the authors apply the techniques they have developed in the study of finitely generated commutative monoids to the computation of the elasticity of any such monoid once one of its presentations is known. We take advantage of this work, as well as some fundamental work developed in the study of sets of lengths of factorizations by Geroldinger (summarized in [6]), to provide an algorithm for the computation of the elasticity of any Krull monoid S with finitely generated class group where the set of classes in the divisor class group of S which contain prime divisors is finite. That the elasticity of such a Krull monoid is rational and accepted (i.e., if $\rho(S) = m/n$, then there are irreducibles $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m$ with $\alpha_1 \cdots \alpha_n = \beta_1 \cdots \beta_m$) follows from the main theorem of [3] and will also follow as a consequence of Theorem 6. While several of the papers mentioned above offer specific calculations of $\rho(S)$ in the Krull case, nowhere is a general method to compute such an elasticity outlined. In fact, in a recent paper [5] Chapman and D.F. Anderson explore the question of determining the complete set of elasticities of Krull domains with given finite cyclic divisor class group. Calculations in [5] are restricted to the special case where the cross number (see [6]) of the divisor class group is 1.

We follow our Introduction with a short summary which outlines some basic definitions, the technical construction of a Krull monoid, the necessary definitions for the algorithm from the study of zero-sum problems, and some basic facts concerning the elasticity of a Krull monoid. Section 3 contains the actual algorithm illustrated by several examples. Our algorithm relies on the computation of the minimal nontrivial solutions of a finite system of linear Diophantine equations, which is itself an actual research area. As techniques in this area improve, so will the efficiency of this algorithm.

2. Krull monoids, minimal zero-sequences and elasticity

We assume that all our monoids are commutative and cancellative ($a \cdot c = a \cdot b$ implies $c = b$). If S is such a monoid written multiplicatively, then $s \in S$ is a *unit* if there exists $s' \in S$ with $ss' = 1$. Let $\mathscr{U}(S)$ denote the set of units of S. If $\mathscr{U}(S) = \{1\}$, then S is called *reduced*. If a, $b \in S$, then we say that *a divides b* (denoted $a|b$) if there exists $s \in S$ with $sa = b$. Two elements a, $b \in S$ are *associates* if there is

$s \in \mathscr{U}(S)$ such that $sa = b$ and we denote this relationship by $a \simeq b$. An element $s \in S$ is *irreducible* (or an *atom*) in S if $a \mid s$ for $a \in S$ implies that either $a \in \mathscr{U}(S)$ or $a \simeq s$. As previously mentioned, let $\mathscr{A}(S)$ represent the set of atoms of S. If $\alpha_1, \ldots, \alpha_k$ are nonassociate elements of $\mathscr{A}(S)$ and $n_1, \ldots, n_k$ are positive integers, then $a = u \prod_{i=1}^{k} \alpha_i^{n_i}$, where $u \in \mathscr{U}(S)$, is a *decomposition* of a in S. Two decompositions $u \prod_{i=1}^{k} \alpha_i^{n_i} = v \prod_{j=1}^{t} \beta_j^{m_i}$ are *associated* if $k = t$ and for some permutation σ of $\{1, \ldots, k\}$, $\alpha_i \simeq \beta_{\sigma(i)}$ and $n_i = m_{\sigma(i)}$ for each i. A *factorization* of $a \in S$ is an equivalence class of associated decompositions.

A monoid S is a *Krull monoid* if there exists a free Abelian monoid D and a homomorphism $\partial : S \to D$ such that
1. $x \mid y$ in S if and only if $\partial(x) \mid \partial(y)$ in D,
2. every $\beta \in D$ is the greatest common divisor of some set of elements in $\partial(S)$.

The basis elements of D are called the *prime divisors* of S and the quotient $D/\partial(S)$ is called the *divisor class group of S* and denoted $\mathrm{Cl}(S)$. If S is a Krull monoid and x is a nonunit of S, then there exist a unique set $p_1, \ldots, p_k$ of prime divisors of S and unique positive integers $n_1, \ldots, n_k$ such that

$$\partial(x) = p_1^{n_1} \cdots p_k^{n_k}. \tag{1}$$

Notice (1) implies that $n_1 p_1 + \cdots + n_k p_k = 0$ in $\mathrm{Cl}(S)$. The interested reader can find more information on Krull monoids in [11].

If S is an atomic monoid and x a nonunit of S, then set

$$L(x) = \left\{ n \;\middle|\; \text{there exist } \alpha_1, \ldots, \alpha_n \in \mathscr{A}(S) \text{ with } x = \alpha_1 \cdots \alpha_n \right\}$$

and

$$\mathscr{L}(S) = \left\{ L(x) \;\middle|\; x \text{ a nonunit of } S \right\}.$$

By the above discussion, if S is a Krull monoid and x a nonunit of S, then the structure of the set $L(x)$ (and hence $\mathscr{L}(S)$) is dependent on the behavior of finite sequences of elements in $\mathrm{Cl}(S)$ which sum to zero. We will require the following definitions. Let G be a finitely generated Abelian group and let $T = \{g_1, \ldots, g_t\}$ be a system of generators of G. An element $(n_1, \ldots, n_t) \subseteq \mathbb{N}^t$ is a *zero-sequence* in G of elements from T if $\sum_{i=1}^{t} n_i g_i = 0$. We denote by $\mathscr{B}(G, T)$ the set of zero-sequences of elements from T. If $x = (x_1, \ldots, x_t)$ and $y = (y_1, \ldots, y_t)$ are in $\mathscr{B}(G, T)$, then $\mathscr{B}(G, T)$ forms a commutative cancellative monoid under the operation

$$x + y = (x_1 + y_1, \ldots, x_t + y_t).$$

$\mathscr{B}(G, T)$ is known as *the block monoid of G* over T (more information on block monoids can be found in [10]). A zero-sequence $(n_1, \ldots, n_t)$ is *minimal* provided that there is no other zero-sequence $(n_1', \ldots, n_t')$ such that $(n_1', \ldots, n_t') < (n_1, \ldots, n_t)$ (here we are using the usual partial ordering on $\mathbb{N}^t$). If $\mathscr{M}(G, T)$ is the set of minimal zero-sequences of $\mathscr{B}(G, T)$, then $\mathscr{M}(G, T) = \mathscr{A}(\mathscr{B}(G, T))$. The length of the longest minimal zero-sequence of $\mathscr{M}(G, T)$ is called *the Davenport Constant of G with respect to T* and denoted $D_T(G)$.

Proposition 1. *Let S be a Krull monoid with finitely generated divisor class group G such that the set T of divisor classes of G which contain prime divisors is finite.*
1. [6, Lemma 4.2] $\mathscr{L}(S) = \mathscr{L}(\mathscr{B}(G,T))$ *and hence* $\rho(S) = \rho(\mathscr{B}(G,T))$.
2. [5, Proposition 3] $1 \leqslant \rho(\mathscr{B}(G,T)) \leqslant \frac{1}{2}D_T(G)$.
3. [5, Proposition 3] $\rho(\mathscr{B}(G,T)) = \frac{1}{2}D_T(G)$ *if and only if there exists a minimal zero sequence* $(g_1,\ldots,g_{D_T(G)})$ *in* $\mathscr{M}(G,T)$ *with* $-g_i \in T$ *for each* $1 \leqslant i \leqslant D_T(G)$.

Hence, to compute the elasticity of a Krull monoid S satisfying the hypothesis of Proposition 1, we need only consider the appropriate block monoid. For this reason, we will use additive notation for our monoids throughout the rest of this paper.

3. Elasticity of full affine semigroups

A monoid S is a *full affine semigroup* if there exists $p \in \mathbb{N}$ and N a subgroup of $\mathbb{Z}^p$ such that $S = N \cap \mathbb{N}^p$. It can be easily proven that if S is nontrivial, then it is generated by the set of minimal elements of $(N \cap \mathbb{N}^p) \setminus \{0\}$ with respect to the usual partial order on $\mathbb{N}^p$. By Dickson's Lemma, this set of minimal elements is always finite and thus every full affine semigroup is a finitely generated monoid. These monoids are always reduced (they have no units but the zero element) since they are included on $\mathbb{N}^p$ for some p, and are also cancellative for the same reason. We show next that they are also atomic.

Lemma 2. *Let S be a full monoid and take* $a, b \in S$. *Then* $a \simeq b$ *if and only if* $a = b$.

Proof. This follows easily from the fact that S is cancellative and reduced. □

Lemma 3. *Let S be a full monoid and let* $s \in S$. *The following conditions are equivalent*:
1. $s \in \mathscr{A}(S)$.
2. *If* $s = a + b$ *for some* $a, b \in S$, *then either* $a = 0$ *or* $b = 0$.

Proof. Follows from the definition of irreducible element and the fact that S is reduced and cancellative. □

Proposition 4. *Let S be a nontrivial full affine semigroup. Then* $\mathrm{Minimals}_{\leqslant}(S\backslash\{0\}) = \mathscr{A}(S)$. *In particular, this implies that S is atomic.*

Proof. Assume that $S = N \cap \mathbb{N}^p$ for some subgroup N of $\mathbb{Z}^p$.

Take $m \in \mathrm{Minimals}_{\leqslant}(S\backslash\{0\})$. Then by the minimality of m and Lemma 3 we get that $m \in \mathscr{A}(S)$.

Now take $a \in \mathscr{A}(S)$. Then there exists $m \in \text{Minimals}_{\leqslant}(S\backslash\{0\})$ such that $m \leqslant a$, or in other words, $a = m + x$ for some $x \in \mathbb{N}^p$. Hence $x = a - m \in N \cap \mathbb{N}^p = S$. By Lemma 3 we get that either $m = 0$ or $x = 0$, and since S is nontrivial, $x = 0$. □

As a consequence of these results, we get that for a full affine semigroup the concepts of decomposition and factorization are the same.

Assume that S is a full affine semigroup and that $\text{Minimals}_{\leqslant}(S\backslash\{0\}) = \{m_1, \ldots, m_s\}$. Define

$$\varphi : \mathbb{N}^s \to S, \quad \varphi(a_1, \ldots, a_s) = \sum_{i=1}^{s} a_i m_i.$$

Observe that if $x \in S$ and we have a factorization of x of the form $x = \sum_{i=1}^{s} a_i m_i$, then $(a_1, \ldots, a_s) \in \varphi^{-1}(x)$. Note as well that $(a_1, \ldots, a_s)$, $(b_1, \ldots, b_s)$ belong to the kernel congruence of φ if and only if $\sum_{i=1}^{s} a_i m_i = \sum_{i=1}^{s} b_i m_i$, which holds if and only if $(a_1 - b_1, \ldots, a_s - b_s)$ belongs to the subgroup M of $\mathbb{Z}^s$ given by the equations

$$M \equiv m_1 x_1 + \cdots + m_s x_s = 0.$$

Hence

$$\text{Ker}(\varphi) = \sim_M = \left\{ (a, b) \in \mathbb{N}^s \times \mathbb{N}^s \,\middle|\, a - b \in M \right\},$$

and S is isomorphic to $\mathbb{N}^s / \sim_M$. For a given element $a = (a_1, \ldots, a_s) \in \mathbb{N}^s$, set $|a| = \sum_{i=1}^{s} a_i$. Further, for $x \in S \setminus \{0\}$, set

$$\rho(x) = \sup \left\{ \frac{n}{m} \,\middle|\, \begin{array}{l} x = \alpha_1 + \cdots + \alpha_n = \beta_1 + \cdots + \beta_m \\ \text{where each } \alpha_i \text{ and } \beta_j \in \mathscr{A}(S) \end{array} \right\}.$$

Using all this information, we get the following result.

Lemma 5. *Let S be a full affine semigroup and let M be as above.*

1. *For every* $x \in S\backslash\{0\}$,

$$\rho(x) = \sup \left\{ \frac{|a|}{|b|} \,\middle|\, a, b \in \varphi^{-1}(x) \right\}.$$

2. $$\rho(S) = \sup \left\{ \frac{|a|}{|b|} \,\middle|\, (a, b) \in \sim_M \backslash\{(0, 0)\} \right\}.$$

The supremum taken in the first statement of Lemma 5 can be replaced by a maximum, since it can be shown in this setting that the set $\varphi^{-1}(x)$ is finite.

In addition, the congruence $\sim_M$ is itself a submonoid of $\mathbb{N}^s \times \mathbb{N}^s$ that is generated by the set of nonnegative minimal nonzero solutions of the system of equations

$$m_1x_1 + \cdots + m_sx_s - m_1y_1 - \cdots - m_sy_s = 0$$

(see [15] for details), which is a finite set. Let us denote this set by $\mathscr{I}(\sim_M)$ (this set is in fact the set of atoms of $\sim_M$ as a monoid). The following result tells us that for computing the elasticity of S we do not have to compute $|a|/|b|$ for all $(a, b) \in \sim_M \setminus\{(0, 0)\}$; it suffices to compute these values for the elements of $\mathscr{I}(\sim_M)$.

Theorem 6. *Let S be a full affine semigroup and let*

$$\{m_1, \ldots, m_s\} = \text{Minimals}_{\leqslant}(S \setminus \{0\}).$$

Take M to be the subgroup of $\mathbb{Z}^s$ *defined by the equations* $m_1x_1 + \cdots + m_sx_s = 0$. *Then*

$$\rho(S) = \max\left\{ \frac{|a|}{|b|} \,\middle|\, (a, b) \in \mathscr{I}(\sim_M)\right\}.$$

Proof. Since every $(a, b) \in \mathscr{I}(\sim_M)$ is in $\sim_M$, we have by Lemma 5 that

$$\max\left\{|a|/|b| \,\middle|\, (a, b) \in \mathscr{I}(\sim_M)\right\} \leqslant \rho(S).$$

We now prove for every $(x, y) \in \sim_M \setminus\{(0, 0)\}$ that we have $|x|/|y| \leqslant |a|/|b|$ for some $(a, b) \in \mathscr{I}(\sim_M)$, which in view of Lemma 5 proves the other inequality.

As $\mathscr{I}(\sim_M)$ is a system of generators of $\sim_M$ as a monoid, there exist elements $(a_1, b_1), \ldots, (a_t, b_t) \in \mathscr{I}(\sim_M)$ such that $(x, y) = \sum_{i=1}^{t}(a_i, b_i)$, whence

$$|x|/|y| = \left(\sum_{i=1}^{t} |a_i|\right) \Big/ \left(\sum_{i=1}^{t} |b_i|\right).$$

The proof follows from the easy fact that for $k_1, k_2, l_1, l_2 \in \mathbb{N}$ one has that $(k_1 + k_2)/(l_1 + l_2) \leqslant \max\{k_1/l_1, k_2/l_2\}$. □

A more general version of Theorem 6 can be found in [16].

If we are given a full affine semigroup $S = N \cap \mathbb{Z}^p$, then the set $\text{Minimals}_{\leqslant}(S \setminus \{0\})$ can be computed by using the procedure described in [9] for finding the minimal nonnegative nontrivial solutions of a system of linear Diophantine equations, or using the results given in [15] or [14]. Furthermore, once we have computed $\text{Minimals}_{\leqslant}(S \setminus \{0\})$, the set of elements $\mathscr{I}(\sim_M)$, with M as described above, is computed by using the same method. Once we have all this information, the elasticity of S is calculated using Theorem 6.

3.1. The elasticity of Krull monoids

Let G be a finitely generated Abelian group and let $T = \{g_1, \ldots, g_t\}$ be a system of generators of G. Assume that $G = \mathbb{Z}^n \times \mathbb{Z}_{d_1} \times \cdots \times \mathbb{Z}_{d_r}$ and set N to be the subgroup of $\mathbb{Z}^t$ whose defining equations are

$$
\begin{array}{lcll}
a_{11}x_1+\cdots+a_{1t}x_t & = & 0, & \\
 & \vdots & & \\
a_{n1}x_1+\cdots+a_{nt}x_t & = & 0, & \\
a_{(n+1)1}x_1+\cdots+a_{(n+1)t}x_t & \equiv & 0 & \pmod{d_1}, \\
 & \vdots & & \\
a_{(n+r)1}x_1+\cdots+a_{(n+r)t}x_t & \equiv & 0 & \pmod{d_r},
\end{array}
$$

where $(a_{1i},\dots,a_{(n+r)i})=g_i$. Then $\mathscr{B}(G,T)$ is just the monoid of nonnegative integer solutions of the above system of equations. That is, $\mathscr{B}(G,T)=N\cap\mathbb{N}^t$ and $\mathscr{M}(G,T)=\mathrm{Minimals}_{\leqslant}((N\cap\mathbb{N}^p)\backslash\{0\})$. Assume that $\mathscr{M}(G,T)=\{m_1,\dots,m_s\}$. Then $\mathscr{B}(G,T)=\langle m_1,\dots,m_s\rangle$ and is isomorphic to $\mathbb{N}^s/\sim_M$, where M is the subgroup of $\mathbb{Z}^s$ with defining equations

$$
(m_1\cdots m_s)\begin{pmatrix}x_1\\ \vdots\\ x_s\end{pmatrix}=0.
$$

In view of Theorem 6, in order to compute the elasticity of $\mathscr{B}(G,T)$, we only have to compute $\mathscr{I}(\sim_M)$ and then

$$
\rho(\mathscr{B}(G,T))=\max\left\{\left.\frac{|a|}{|b|}\right|(a,b)\in\mathscr{I}(\sim_M)\right\}.
$$

Example 7. Let $G=\mathbb{Z}_6$ and let $T=\{\overline{1},\overline{2}\}$. Then N is the subgroup of $\mathbb{Z}^2$ with defining equation

$$
x_1+2x_2\equiv 0 \pmod{6}.
$$

Further $\mathscr{B}(G,T)=N\cap\mathbb{N}^2=\langle(0,3)(2,2)(4,1)(6,0)\rangle\cong\mathbb{N}^4/\sim_M$, where M is the subgroup of $\mathbb{Z}^4$ with defining equations

$$
\begin{pmatrix}0&2&4&6\\3&2&1&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=0.
$$

Computing $\mathscr{A}(\sim_M)$ we obtain

$$
\begin{aligned}
\mathscr{A}(\sim_M)=\big\{&(e_1,e_1),(e_2,e_2),(e_3,e_3),(e_4,e_4),\\
&((0,0,2,0),(0,1,0,1)),((0,0,3,0),(1,0,0,2)),\\
&((0,1,0,1),(0,0,2,0)),((0,1,1,0),(1,0,0,1)),\\
&((0,3,0,0),(2,0,0,1)),((0,2,0,0),(1,0,1,0)),\\
&((1,0,0,2),(0,0,3,0)),((1,0,0,1),(0,1,1,0)),\\
&((2,0,0,1),(0,3,0,0)),((1,0,1,0),(0,2,0,0))\big\}
\end{aligned}
$$

where e_i is the element whose ith coordinate is 1 and all other coordinates zero.

Therefore

$$\rho(\mathscr{B}(G,T)) = \max\{1\} = 1$$

and thus $\mathscr{B}(G,T)$ is half-factorial (that $\rho(\mathscr{B}(G,T)) = 1$ also follows from [7, Theorem 3.8]).

Example 8. Now take $G = \mathbb{Z}_6$ and $T = \{\overline{1}, \overline{4}\}$. In this case N is the subgroup of $\mathbb{Z}^3$ with defining equation

$$x_1 + 4x_2 \equiv 0 \pmod 6.$$

Here $\mathscr{M}(G,T) = \{(0,3), (2,1), (6,0)\}$ and thus $\mathscr{B}(G,T)$ is isomorphic to $\mathbb{N}^4/\sim_M$, where M is defined by the equations

$$\begin{pmatrix} 0 & 2 & 6 \\ 3 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

Calculating the set $\mathscr{I}(\sim_M)$ we obtain

$$\mathscr{I}(\sim_M) = \big\{((0,0,1),(0,0,1)), ((0,1,0),(0,1,0)), ((0,3,0),(1,0,1)), \\ ((1,0,1),(0,3,0)), ((1,0,0),(1,0,0))\big\},$$

whence $\rho(\mathscr{B}(G,T)) = \rho([(1,0,1)]) = \rho((6,3)) = \frac{3}{2}$. The reader should note here in light of Proposition 1 that

$$1 < \rho(\mathscr{B}(G,T)) < \tfrac{1}{2} D_T(\mathbb{Z}_6) = 3.$$

3.2. The elasticity of Diophantine monoids

A monoid S is a *Diophantine monoid* if it is a full affine semigroup $N \cap \mathbb{N}^p$ for some $p \in \mathbb{N}$ where N is a subgroup of $\mathbb{Z}^p$ such that the invariant factors of N are all equal to one. In other words, S can be defined as the set of nonnegative solutions of a system of linear Diophantine equations. In general, full monoids are not Diophantine (an example of a full monoid that is not Diophantine is given in Example 7). Since every Diophantine monoid is a full affine monoid, we can use the procedure outlined above to compute its elasticity.

Example 9. Let S be the Diophantine monoid given by the equations

$$x_1 + x_2 + x_5 - 2x_6 = 0,$$
$$x_3 + x_4 + x_5 - 2x_7 = 0.$$

Then

$$\mathscr{A}(S) = \{(0,0,0,0,2,1,1), (0,0,0,2,0,0,1), (0,0,1,1,0,0,1), (0,0,2,0,0,0,1), (0,1,0,1,1,1,1), (0,1,1,0,1,1,1), (0,2,0,0,0,1,0), (1,0,0,1,1,1,1), (1,0,1,0,1,1,1), (1,1,0,0,0,1,0), (2,0,0,0,0,1,0)\}$$

and therefore $S = \langle \mathscr{A}(S) \rangle$ is isomorphic to $\mathbb{N}^{11}/\sim_M$, where M is the subgroup of $\mathbb{Z}^{11}$ whose defining equations are

$$\begin{pmatrix} 0&0&0&0&0&0&0&1&1&1&2\\ 0&0&0&0&1&1&2&0&0&1&0\\ 0&0&1&2&0&1&0&0&1&0&0\\ 0&2&1&0&1&0&0&1&0&0&0\\ 2&0&0&0&1&1&0&1&1&0&0\\ 1&0&0&0&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&0&1&1&0&0 \end{pmatrix} \begin{pmatrix} x_1\\ \vdots \\ x_{11} \end{pmatrix} = 0.$$

Computing $\mathscr{I}(\sim_M)$ one obtains

$$\rho(S) = \rho([(0,0,0,0,0,1,0,0,1,0,0)]) = \rho((1,1,2,0,2,2,2)) = \tfrac{3}{2}.$$

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